Beliefs Driven Secular Stagnation

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Abstract

This paper explores the secular stagnation and its implications on asset pricing. I study an endogenous growth model with linear production function. A key feature in my model is the endogenised technology scale factor. It is assumed to be a function of investment capital ratio. With this assumption, the model shows multiplicity. Different interactions between technology and investment will generate different balanced growth paths (BGPs). I also introduce the sunspots. They represent agents' beliefs and serve as the selecting devices. With proper defined Markovian sunspots, the economy sways between different BGPs and growth rates. According to the model, the stagnated growth might be the result of this endogenous system and the pessimistic expectation. Further, this framework can generate the counter-cyclical behaviour of risk premium which puzzles the traditional RBC model.

Key Words: AK model, Secular stagnation, Sunspots, Risk free rate, Risk returns

JEL Codes: E22, E23, G10, G12

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1 Introduction

Macroeconomists seem to have no consensus on the reason why we recover unexpectedly slow from the crisis. Since we observe the slump in Japan in recent decades and slow recoveries in Europe and US after 2008, there are increasing discussions on the concept of "secular stagnation" introduced by Hansen (1939). Differing from conventional business cycle model which predicts the economy will always recover to the "natural" level of growth, the term "secular stagnation" is used to describe the long-lasting stagnated growth. However, there are debates on what are the possible sources of it. The supply side views of Gordon (2015) argue that the primary sources are the innovation pace slowing down and the decrease of labour force participation rate. When we compare productivity change brought by three major industrial revolutions, the paper concludes that the technology development pace is slowing down. Nonetheless, Summers (2014) focuses on the demand side. He argues the reason for slow growth is that the potential "natural" interest rate is lower than the zero lower bound and therefore not feasible. The further problem is that the full employment will never be achievable as well.

I build an endogenous growth model which incorporates two sides. The core concept in the model is the assumption of direct interaction between technology and investment. With this endogenised technology process, the supply side links to demand side. A demand shock on investment will affect the technology and further influence the whole system in the future period. Technology is crucial in the model since it determines the optimal investment level and further the economic growth rate. Moreover, the technology will endogenously responses to investment. I introduce sunspots to capture the demand shock. With these ingredients, the model generates multiple BGPs. The economy follows a process which shifts between different BGPs and growth rates. Importantly, my model produces implications on the asset pricing field that risk premium can have counter-cyclical patterns. To sum, roughly, my model can generate a dynamic system with three endogenous variables and achieves 1) long lasting slow growth and depressing investment, 2) persistent fall in total factor productivity (TFP) and 3) trapped risk-free rate near to the lower bound.

My model is parsimonious and takes a general equilibrium approach. The firm makes the decision on investment to maximise its stock price subject to capital accumulation condition. Importantly, the firm faces an adjustment cost of the investment. The adjustment cost for investment is another crucial assumption in the model. Without adjustment cost, the firm's value will linearly shadow the capital movement which is less interesting. The optimal choice of investment pins down the balanced growth path (BGP) in the economy. On the other hand, the household has the standard problem of maximising utility. As usual, under the complete market and no arbitrage condition, the household's problem determines the unique discount factor to price assets. The linear production "AK" framework ensures the endogenous growth of the firms. Based on this constant return to scale production, I follow Hayashi (1982) in the setups which provide that average "Tobin's Q" will equal to the marginal "q" in the model. Therefore I can conveniently derive the dynamics of firms value and the asset return. Further to the baseline model, an assumption of a simple relation between the technology scale factor A and investment I is made. The technology is a discontinuous function of investment-capital ratio. The inspiration is from the "threshold" assumption in Azariadis & Drazen (1990). Technology will jump to a new level when it reaches the "threshold". This assumption generates multiple BGPs. Moreover, I add sunspots to the economy. The sunspots offer a device which enables the shift to cross multiple equilibria from time to time. The transition is governed by the probability distribution of Markovian sunspots process.

Additionally, this model generates all kinds of patterns of risk premium including the counter-cyclical movement as shown in the data. The empirical evidence for risk premium is that they are large and volatile as surveyed in Cochrane (2005). Figure 1.1 shows the overall trend of the risk premium and risk free rate in the US in recent decades. The dark bar at the bottom shows the risk free rate. After the recent crisis, it drop dramatically and nearly hit the zero lower bound. The grey area is the risk premium. It expands accordingly since there is no much change in the risk asset return. However, the standard real business cycle (RBC) theorem failed to predict that. Especially for risk-free rate, from the Euler equation, the risk-free rate is determined by expectation of the intertemporal rate of substitution in marginal utility. Since the growth of consumption is relatively stable in the observation, the conventional RBC model cannot explain the "excess" volatility of the risk-free rate data. Among many others, Gourio (2012) and Gabaix (2012) consider modelling massive "disaster" shock in the RBC model and successfully generate the time-varying risk premium. My result is similar to theirs in term of the possibility of a large downward shift in economy growth. Likewise, my model can account for the historical movement of risk premium and risk-free rate.

One closely related paper is Benigno & Fornaro (2016). They build a new Keynesian growth model with wage rigidities and vertical innovation. Our works are similar in the sense that investment directly co-move with productivity and it further generates multiple balanced growth paths (BGPs). Additionally, sunspot plays the role of the selection device. However, in their work, the two equilibria of the economy are derived by setting nominal interest rate to 0 or employment to full state while mine has the mechanism to account the related movement of risk-free rate and risk premium.

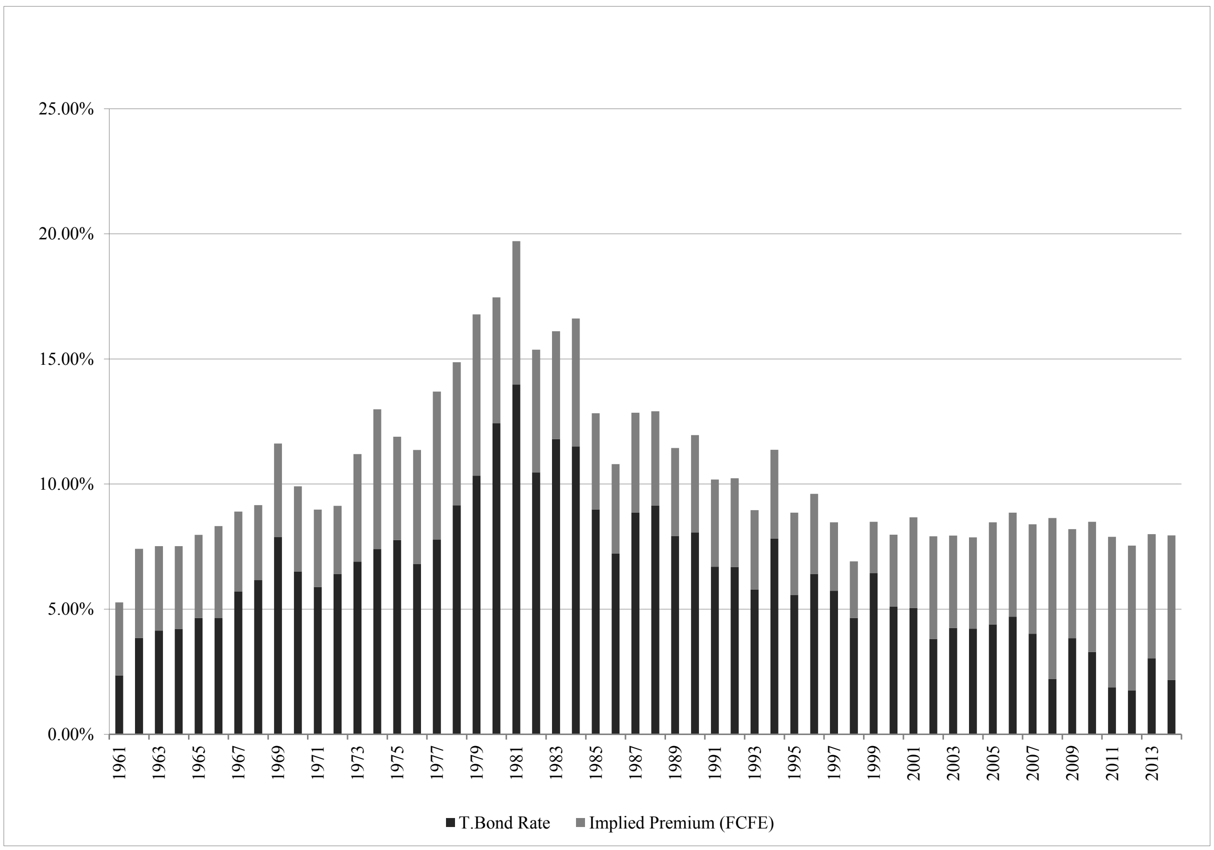

Figure 1.1: Historical Risk Premium and Risk Free Rate in US

The study is built on two main fields of the literature. Firstly, there are studies on firms investment behaviour such as Cochrane (1991), Jermann (1998), Boldrin et al. (2001) and Campbell (2003). These studies pay attention to relations among firms investments, stock returns and macroeconomic fluctuations. The overall general equilibrium structure in this paper is following literature in this area. The equilibrium exists across the consumer side and firm side. While the firm maximising the stock value and consumer maximising the utility, assets issued by firm and held by consumer is the linkage between the two. In equilibrium, at the aggregate level, the investor holds all the stock and consumes the dividend. The consumer want a smooth consumption. The willingness of sacrifices of today's consumption for tomorrow will offer a price to the asset. Observing this pricing kernel, the firm makes decision on how much profit should be retained as investment for the value maximisation goal. Kogan & Papanikolaou (2012) survey research in this field. Regarding methodology, more closely related papers are Kogan (2001) and Eberly & Wang (2009). They solve the central planner's problem and competitive equilibrium in continuous time version similar to my model. They derive the closed-form solution for the optimal condition of the problem.

Second related field concerns the models dealing with endogenous growth as in Fatas (2000) and Azariadis & Drazen (1990) among many others. The former considers the AK framework with cyclical shocks. It is, in some sense, a version of my model without investment adjustment cost. However, the aim for the paper is to explain the persistent fluctuations in output. The latter is the inspiration for us about the assumption made on the relation between technology and investment. In terms of growth and sunspots, it is not rare to use the different structure of production function, non-linear accumulation of capital and extrinsic randomness to handle multiple equilibria and indeterminacy as surveyed in Benhabib & Farmer (1999).

The paper is composed as follows. Section 2 sets the baseline model and solution in a deterministic fashion. In section 3, I introduce the fundamental assumption which endogenises the technology. Sector 4 presents the sunspots and illustrates the implication of the model in terms of growth and asset pricing. Further, section 5 runs calibration and indicate the limit and further direction of the research. Section 6 concludes the paper.

2 Baseline Model and Solution

In this section, I solve a baseline model to establish the relation between firm's investment behaviour and technology. The model describes a production economy with one productive sector. Time t discretely runs from 0 to infinity. There are a large number of identical firms and consumers in the economy. I follow the framework used by Christiano & Harrison (1999) and consider a state variable s_t. Let $s^t = (s_0, s_1, \ldots, s_t)$ be notation of the history of the state variable. The probability of history s^t is denoted by $\mu(s^t)$. All endogenous variables introduced later are functions of histories s^t. The production function is linear as $Y(s^t) = A(s^t)K(s^t)$. Y, K and A denote the output, capital stock and exogenous technology scale factor respectively. Firm uses operation profit to pay the dividends as $D(s^t) \equiv A(s^t)K(s^t) - I(s^t)$, where I is the investment. The firm maximises its stock value through discounted cash flow,

$$V\left(s^t\right) = \underset{I}{Max} \quad \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \mu(s^\tau) \beta^{\tau-t} \frac{\Lambda(s^\tau)}{\Lambda(s^t)} D(s^\tau) \tag{2.1}$$

subject to the constrains

$$\frac{K(s^{t+1})}{K(s^t)} = 1 - \delta + \phi\left(i\left(s^t\right)\right) \tag{2.2}$$

$$A\left(s^t\right) > 0, \quad K\left(s^t\right) > 0, \quad \Lambda\left(s^t\right) > 0 \tag{2.3}$$

$$V\left(s^t\right) > 0, \quad I\left(s^t\right) > 0 \tag{2.4}$$

$$K\left(s^0\right) \textit{ is given.} \tag{2.5}$$

Where $\beta \in (0, 1)$ is the time preference parameter. β and Λ together constitute the discount factor. I assume the firms are price takers. The discount factor here is exogenously taken as given. The equation (2.2) is the capital accumulation condition. $\delta \in (0, 1)$ is the capital depreciation rate and $i \equiv I/K$. The function $\phi(\cdot)$ captures the effectiveness in converting investment to capital inputs. For the convenience of later referring, I slightly abuse the terminology and name this function as investment adjustment cost function. If $\phi(i) = i$, the capital accumulation condition becomes $K_{t+1} = (1-\delta)K_t + I_t$, there is no adjustment cost for investment. However, I constrain the function to $\phi(i) > 0$, $1 > \phi'(i) > 0$ and $\phi''(i) \leq 0$ to capture the convexity of adjustment cost which follows the convention in the literature.[1] The more investment the firm devote, the more costly it is. I assume the function is homogeneous of degree one in I and K to follow the proposition of Hayashi (1982). This proposition makes the model easy to tackle and helps to derive the findings in asset pricing.

There is a broad range of literature discuss the linear production, constant return to scale and externality. Here I borrow the intuition used in Azariadis & Drazen (1990). There is

[1] See appendix A.

a distinction between the *private* and *public* factors in production as introduced by Romer (1986). The private factor is controlled by individual firms. The public factor is not controlled by any specific producer. In the production process, there are spillovers from the private capital stock. The spillovers contribute to the public factor. In the aggregate level, total factor productivity A consists of both two factors. Because of the externality, the production in the economy shows constant return to scale.

I assume the stocks issued by firms are the only kind of assets existing in the economy and normalise the aggregate shares to unity. The representative consumer faces a standard infinite horizon utility maximisation problem given by,

$$J\left(s^t\right) = \underset{C}{Max} \quad \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \mu\left(s^\tau\right) \beta^{\tau-t} U\left(C\left(s^\tau\right)\right) \tag{2.6}$$

subject to budget constrain

$$S\left(s^{t+1}\right) P\left(s^t\right) = S\left(s^t\right) \left[P\left(s^t\right) + D\left(s^t\right)\right] - C\left(s^t\right) \tag{2.7}$$

where S is the stock shares holding by the consumers. P is the asset price. C is the consumption.

The model is closed by the resource constrain,

$$C\left(s^t\right) = D\left(s^t\right) \tag{2.8}$$

From the consumer side, the model is a standard consumption-based capital asset pricing model (CCAPM) which can be dated back to Lucas (1978). In equilibrium, the representative investor holds the single asset and consumes its dividend. As shown later, the price the asset is related to the marginal rate of substitution of consumption.

2.1 Firm's Optimal Behaviour and General Equilibrium Condition

I use dynamic programming to derive the first order conditions. The firms' problem can be easily written in recursive form. The Bellman equation is

$$V\left(s^t\right) = \max_{I_t} \quad A\left(s^t\right) K\left(s^t\right) - I\left(s^t\right) + \sum_{s^{t+1}|s^t} \mu\left(s^{t+1}\right) \beta \frac{\Lambda\left(s^{t+1}\right)}{\Lambda\left(s^t\right)} V\left(s^{t+1}\right) \tag{2.9}$$

First order condition (FOC) for I_t is,

$$\sum_{s^{t+1}|s^t} \mu\left(s^{t+1}\right)\beta\frac{\Lambda\left(s^{t+1}\right)}{\Lambda\left(s^t\right)}\frac{\partial V\left(s^{t+1}\right)}{\partial K\left(s^{t+1}\right)} = \frac{1}{\phi'\left(i\left(s^t\right)\right)} \tag{2.10}$$

The prime $'$ is used to denote the derivative as $\phi'(i) \equiv \partial\phi(i)/\partial i$ and $\phi''(i) \equiv \partial^2\phi(i)/\partial i^2$. I follow the convention to name the marginal price of capital on the left-hand side as "marginal q". The firm's value is maximised when the investment is chosen to balance the expected marginal gain and marginal lost to the price.

For a problem like this, we have a favourable condition derived by Hayashi (1982) stating that "marginal q" ($\partial V/\partial K$) will equal to the "average Q" (V/K). A Rigorous proof is in appendix B. Therefore, I divide both side of the Bellman equation (2.9) by K_t and substitute the FOC of I (2.10), capital accumulation condition (2.2) and Hayashi proposition in it to obtain,

$$\frac{V\left(s^t\right)}{K\left(s^t\right)} = A\left(s^t\right) - i\left(s^t\right) + \frac{\phi\left(i\left(s^t\right)\right) + 1 - \delta}{\phi'\left(i\left(s^t\right)\right)} \tag{2.11}$$

For the simplicity of the notation, from now on, I use $X\left(s^t\right)$ and X_t interchangeably and the simplified notation X when there is no ambiguity.

$$\frac{V}{K} = A - i + \frac{\phi(i) + 1 - \delta}{\phi'(i)} \tag{2.12}$$

Given the predetermined capital stock K, on the equilibrium of firm's side, the firm's value V dose not necessarily has a monotonic relation with the investment. Firstly, $A - i$ presents the current dividend. High investment tends to decrease the current dividend payment to the investors, therefore making the asset less attractive. Next, it enters into the second term. This term is constructed by the growth rate of the economy over $\phi'(i)$ which is the marginal effectiveness of converting the investment into capital. One can check that the second term is positively related to investment-capital ratio i. As a result, there is a trade-off between two channels. An economy with higher i clearly will have higher growth yet not necessarily will have higher price of the asset. However the implication for the asset return is not clear until we derive the completed equilibrium conditions.

Combine (2.10) and (2.11), we have,

$$\sum_{s^{t+1}} \mu\left(s^{t+1} \middle| s^t\right) \beta \frac{\Lambda\left(s^{t+1}\right)}{\Lambda\left(s^t\right)} \left[A\left(s^{t+1}\right) - i\left(s^{t+1}\right) + \frac{\phi\left(i\left(s^{t+1}\right)\right) + 1 - \delta}{\phi'\left(i\left(s^{t+1}\right)\right)}\right] = \frac{1}{\phi'\left(i\left(s^t\right)\right)} \tag{2.13}$$

This is the stochastic difference equation defines the path of the firm's optimal investment behaviour given the process of discount factor Λ and technology A.

The consumer's problem is standard and provides the well-known Euler equation given by [2]

$$P\left(s^t\right) U'\left(C\left(s^t\right)\right) = \sum_{s^{t+1}} \mu\left(s^{t+1} \middle| s^t\right) \beta U'\left(C\left(s^{t+1}\right)\right) \left[P\left(s^{t+1}\right) + D\left(s^{t+1}\right)\right] \tag{2.14}$$

Euler equation offers the discount factor that $\Lambda = U'(C)$.

Now, we are ready to define an equilibrium for the modelled economy.

Definition 1. *An equilibrium is a set of sequences* $K^*\left(s^t\right)$, $I^*\left(s^t\right)$, $i^*\left(s^t\right)$, $D^*\left(s^t\right)$, $C^*\left(s^t\right)$, $S^*\left(s^t\right)$, $A\left(s^t\right)$, $\Lambda\left(s^t\right)$, $V\left(s^t\right)$, *such that:*

1. $C^*\left(s^t\right)$ *and* $S^*\left(s^t\right)$ *solve the household's optimization problem (2.6), given* $V\left(s^t\right)$ *and* $D^*\left(s^t\right)$.

2. $K^*\left(s^t\right)$, $I^*\left(s^t\right)$, $i^*\left(s^t\right)$ *and* $D^*\left(s^t\right)$ *solve the firm's problem (2.1), given* $A\left(s^t\right)$, $\Lambda\left(s^t\right)$ *and initial capital stock in the economy.*

3. $\Lambda\left(s^t\right)$ *is the unique discount factor satisfying* $\Lambda\left(s^t\right) = U'\left(C^*\left(s^t\right)\right)$.

4. *Markets clear:* $C^*\left(s^t\right) = D^*\left(s^t\right)$.

5. *Transversality conditions hold.*[3]

As usual, I combine the optimal conditions on both sides to obtain the general equilibrium condition. For simplicity, this paper considers the log-utility case which makes $\Lambda = C^{-1}$. With the market clearing condition $C = D = AK - I$, we can rearrange equation (2.13) into

$$\beta + \beta E_t \left[\frac{\phi\left(i_{t+1}\right) + 1 - \delta}{\phi'\left(i_{t+1}\right)\left(A_{t+1} - i_{t+1}\right)}\right] = \frac{\phi\left(i_t\right) + 1 - \delta}{\phi'\left(i_t\right)\left(A_t - i_t\right)} \tag{2.15}$$

where E is the expectation operator. This is the core stochastic difference equation to govern the movement of firm's optimal decision. Ideally, with the initial value, the firm will

[2] See appendix C for derivation.

[3] See appendix D for details.

solve this equation recursively to ensure it being on the path which maximises its value. By solving it, we obtain several findings on macroeconomic fundamentals and asset prices in this modelled economy.

Proposition 1. *In the modelled economy, there is no transition dynamics towards equilibrium.*

Proof. One can check that the difference equation (2.15) has one fix point.

$$\frac{\phi(i) + 1 - \delta}{\phi'(i)(A - i)} = \frac{\beta}{1 - \beta} \tag{2.16}$$

□

In some sense, we can say the model is static. Firms will directly choose the optimal i which places the economy on equilibrium according to the observed exogenous A. Even though A is a stochastic process, the decision of i made by the firm is simply according to current realisation of A as the solution of (2.16). [4] Since there is no consensus on what should be the proper functional form for $\phi(\cdot)$, I will not solve the equation (2.16) explicitly, However, for the convenience of later analysis, I use $i[A]$ to denote the solution of a given A. In this situation, i will shadow the movement of the technology process A.

Proposition 2. *If technology scale A is increasing over time, the investment-capital ratio i will increasing to catch up. However, the the distant $A - i$ will be expanded.*

Proof. This comes from applying implicit function theorem to (2.16), which offers

$$\frac{\partial i}{\partial A} = \frac{[\phi(i) + 1 - \delta]\,\phi'(i)}{(\phi'(i))^2 (A - i) - \phi''(i)[\phi(i) + 1 - \delta](A - i) + [\phi(i) + 1 - \delta]\,\phi'(i)} \tag{2.17}$$

With our assumption on $\phi(i)$, $\phi'(i)$, $\phi''(i)$ and $A - i = (D/K) > 0$, we have $1 > \partial i/\partial A > 0$.

□

A higher technology will be corresponded with higher investment by firms to obtain the optimal firm's value. Further, this relation tells that, when A is increasing, the investment-capital ratio will not catch up as same the peace of A since $1 > \partial i/\partial A$. In other words, the dividend payment $D = AK - I$ will increase in the situation of accumulation of technology A.

[4] See appendix E for the discussion of the root.

For the economic growth, it is clear that once A and i are determined the other endogenous variables will all growth at rate $\phi(i) + 1 - \delta$.

Proposition 3. *The risk premium RP, expected risky asset return $E_t(R_{t+1})$ and risk-free rate r^f are given by,*

$$RP\left(s^t\right) = E_t\left(R_{t+1} \middle| s^t\right) - r^f\left(s^t\right) \tag{2.18}$$

$$E_t\left(R_{t+1} \middle| s^t\right) = \frac{1}{1-\beta}\phi'\left(i\left[A_t\right]\right) E_t\left[A_{t+1} - i\left[A_{t+1}\right] \middle| s^t\right] \tag{2.19}$$

$$r^f\left(s^t\right) = \frac{1}{1-\beta}\phi'\left(i\left[A_t\right]\right)\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]} \middle| s^t\right)\right]^{-1} \tag{2.20}$$

Proof. See appendix G. □

When i's are substituted by the root of the equation (2.16), RP is an expression only involving the technology process A. If s^t is independent and identically distributed, high RP corresponds to low i and slow current growth since we have $\phi''(i) < 0$. To elaborate, this model shows that if we currently experience slow growth, we will observe a higher risk premium than it in periods enjoying fast growth. To sum, with the i.i.d assumption, this baseline model can generate counter-cyclical risk premium.

From this aspect, the counter-cyclical risk premium is caused by the adjustment cost of investment. Recall the situation when $\phi(i) \to i$, there is no cost for investment. Accordingly, $\phi'(i) \to 1$. Here, the risk premium becomes

$$RP\left(s^t\right) = \frac{1}{1-\beta}\left\{E_t\left[A_{t+1} - i\left[A_{t+1}\right] \middle| s^t\right] - \left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]} \middle| s^t\right)\right]^{-1}\right\} \tag{2.21}$$

which is a constant under the i.i.d assumption. **[INTUITION?!]**

In fact, there is another major source which drives the behaviour of these asset return terms namely the expectation term. In later sections, I will release the i.i.d assumption and study the term in the expectation operator. If the distribution of A_{t+1} is related to the value of A_t, the movement of risk premium will be affected by the expectation term as well. That will be the second source to account for the counter-cyclical behaviour in data.

3 Endogenous Productivity and Multiple Equilibria

The baseline model shows some conventional logic of the economic growth. The equilibrium investment is decided by firms according to productivity. Further, the level of investment pins down the growth rate. In this section, I endogenise the technology. The inspiration comes from the literature on growth theory and poverty trap. One field of the growth model assumes technology A being a function of capital K or investment I to lay out multiple equilibria. Azariadis & Stachurski (2005) have a good survey on that field. By doing that, it is possible for convex neoclassical growth model to generate multiple BGP's to explain the self-reinforced poverty seen in many developing countries. In addition, there are considerable number of research trying to provide micro-foundation to the direct relation between technology and investment. Roughly speaking the existing micro-foundation for that comes from modelling the imperfect competition or complementarities. Among many others, Matsuyama (1997) has a series of papers discussing the feeding back of investment to externalities and hence productivity. Besides, Da Rin & Hellmann (2002)'s paper emphasises the bank with sufficient market power will act as a coordinator to encourage the entrepreneurs to innovate. However, borrowing the structures in those models will complicate the parsimonious model we have. I leave the micro-foundation aside and direct assume a simple relation between technology and investment as in assumption 3. In essence, I borrow the mathematical structure of the "threshold" from Azariadis & Drazen (1990) to generate multiple BGPs. Explicitly, in assumption 3 there is a discontinuous relation between productivity scale factor A and investment-capital ratio i.

Assumption 1: *Technology scale factor A is a discontinuous function for i given by*

$$A\left(i\left(s^t\right)\right) = \begin{cases} A_H;\ if\ i\left(s^t\right) \geq i^*,\ \text{or equivalently } s^t \geq s^* \\ \\ A_L;\ if\ i\left(s^t\right) < i^*,\ \text{or equivalently } s^t < s^* \end{cases} \tag{3.1}$$

I define i_H and i_L to be the corresponding solution to the (2.16) with A_H and A_L respectively such as $i_H = i\left[A_H\right]$ and $i_L = i\left[A_L\right]$.

The assumption describes a threshold relation between technology and investment. The investment-capital ratio should be maintained above a certain level, here is i^*, to assure a high performance of technology level A_H. Otherwise, the technology will be trapped in a relative low level A_L.

The discontinuous function of technology A ensures two solutions of equation (2.16) namely $i\left[A_H\right]$ and $i\left[A_L\right]$ for given A_H and A_L. Intuitively, the more firm invest, the more spillover to the aggregate public production factor. A higher level of technology factor A, in return, urges the firm invest more to catch up. With the discontinuous function, there are two equilibria namely $\{A_H,\ i_H\}$ and $\{A_L,\ i_L\}$ for firms to select. For instance, if all firms choose the high investment-capital ratio i_H, this will generate a high technology scale A_H. In return, A_H confirms that i_H is the optimal choice for an individual firm. Further, there is a critical point, here is i^*, at which the technology takes off. Since we have endogenised the technology, it is of little interest to distinguish which one pins down the other. However, the co-movement assumed is clear.

With this assumption, my model immediately exhibits multiple BGPs. Importantly, these two BGPs correspond to different growth rates, risk-free rate and risk premium. Obviously, the BGP with low steady investment-capital ratio i_L will grow at a slower pace. Without shocks shifting the economy from the slow growth BGP to fast growth BGP, the model shows it will be trapped in there permanently.

Further, I can compare the risk asset return and risk-free rate between 2 states. We have

$$E_t\left(R_{t+1}\middle|\, i_t = i_L\right) = \frac{1}{1-\beta}\phi'\left(i_L\right) E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle|\, i_t = i_L\right] \tag{3.2}$$

$$E_t\left(R_{t+1}\middle|\, i_t = i_H\right) = \frac{1}{1-\beta}\phi'\left(i_H\right) E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle|\, i_t = i_H\right] \tag{3.3}$$

$$\left[r_t^f\middle|\, i_t = i_L\right] = \frac{1}{1-\beta}\phi'\left(i_L\right)\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle|\, i_t = i_L\right)\right]^{-1} \tag{3.4}$$

$$\left[r_t^f\middle|\, i_t = i_H\right] = \frac{1}{1-\beta}\phi'\left(i_H\right)\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle|\, i_t = i_H\right)\right]^{-1} \tag{3.5}$$

Here, we actually back to the asset pricing side discussed in the end of last section. The model predicts that the slow growth state experiences higher expected risky asset return and risk-free rate. With the i.i.d assumption we have $E_t\left(R_{t+1}\middle|\, i_t = i_L\right) > E_t\left(R_{t+1}\middle|\, i_t = i_H\right)$ and $\left[r^f\middle|\, i_t = i_L\right] > \left[r^f\middle|\, i_t = i_H\right]$. However, these conclusions are not consistent with observations as shown in figure 1.1. If we regard the years after 2008 as the "crisis decade" with slow growth, one can see the risk free rate drop in a significant way in this period. However, as have been mentioned, I do not explore the expectation term yet. In next section, the i.i.d assumption will be relaxed. Ultimately, the risk premium is driven by the interaction between adjustment cost function and the distant between technology A and investment capital ratio i. **[......]**

From now on, I consider s_t being non-fundamental shocks having no effect on preferences or technology. Following the convention, I use the term "sunspots". Intuitively, s^t serves as a selecting device to start the endogenous chain reaction as mentioned above. In next subsection, I introduce the sunspots in a formal way.

4 Sunspots Equilibria

In this sector, I formally introduce sunspots equilibria. As noted previously, there was no transition dynamics in the model. The sunspots I defined below are in different environment compare to the mainstream literature. Among many others, Woodford (1986) explained the existence of a continuum of sunspots equilibria which asymptotically converge to the steady state equilibrium. Here, the sunspots equilibria are not construed near the indeterminate equilibrium. My methods of introducing sunspots are more similar to ways in Benigno & Fornaro (2016) and Christiano & Harrison (1999). I apply the term *regime switching sunspot equilibrium* borrowed from Christiano & Harrison (1999) to name the extrinsic random variable in this section.

The simple structure is the following. I specify s_t as an extrinsic random variable. I define $\{s_t\}_0^\infty$ as a Markov chain with state space $\{s_L,\ s_H\}$ and transition matrix $\boldsymbol{P}_s$ given by

$$\begin{pmatrix} p_L & 1-p_L \\ 1-p_H & p_H \end{pmatrix}$$

For example, if $s_t = s_L$, the probability of shifting to the other state s_H is

$$Prob\,(s_{t+1} = s_H |\, s_t = s_L) = 1 - p_L \tag{4.1}$$

Given a history, s^t, we draw s_{t+1}. The realization of s_{t+1} determines which equilibrium we stay, $\{A_H,\ i_H\}$ or $\{A_L,\ i_L\}$, in period $t+1$. I refer the equilibrium as "normal times" and "disaster times" respectively. Apparently, normal times experience faster growth than disaster times since $\phi\,(i_H) > \phi\,(i_L)$. The terminology is from Gourio (2012). Similarly, in that paper, he discusses two possible states experiencing state shifting situation. Nonetheless, in his paper, two states are assigned to two different technology shock process. Here, two states are defined by two distinct equilibrium.

One can reckon s_t being simply a symbol of beliefs. At the beginning of each period, firms observe a signal s_L or s_H. Accordingly, they choose $\{A_H,\ i_H\}$ or $\{A_L,\ i_L\}$ and the economy will growth at rate $\phi\,(i_H)$ or $\phi\,(i_L)$ respectively. The appearance of s_L or s_H is exogenous and governed by Markov property. With this setup, the model result shows simple pattern. The economy switches between two BGPs. This parsimonious structure can generate all kinds of endogenous growth pattern. An example will be that if we set p_L and p_H for a relatively high value, the economy will have a high probability of staying at disaster times (normal times) for a long time.

For the expected risk return and risk-free rate, we can write

$$E\left[R_{t+1}\middle|\, s_t = s_L\right] = \frac{1}{1-\beta}\phi'\left(i_L\right)\left[AiH - p_L\left(AiH - AiL\right)\right] \tag{4.2}$$

$$E\left[R_{t+1}\middle|\, s_t = s_H\right] = \frac{1}{1-\beta}\phi'\left(i_H\right)\left[AiL + p_H\left(AiH - AiL\right)\right] \tag{4.3}$$

$$\left[r_t^f\middle|\, s_t = s_L\right] = \frac{1}{1-\beta}\phi'\left(i_L\right)\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{4.4}$$

$$\left[r_t^f\middle|\, s_t = s_H\right] = \frac{1}{1-\beta}\phi'\left(i_H\right)\left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{4.5}$$

where $AiL \equiv A_L - i_L$ and $AiH \equiv A_H - i_H$.

If p_L equals $1-p_H$, we back to the i.i.d sunspots case. The term in the bracket in the above two expressions will equal to each other. More importantly, if we relax the i.i.d assumption, these Markovian sunspots can generate different kinds of risk premiums pattern by different specification of the transition matrix.

With our assumptions, we have $\phi'\left(i_L\right) > \phi'\left(i_H\right)$ and $AiH > AiL > 0$.[5] Simple calculation tells that, as p_L is increasing, the last terms in the (4.2) and (4.4) are decreasing. For p_H, the opposite is true in (4.3) and (4.5). For efficient large p_L and p_H, it is possible to generate the situation which obtains pro-cyclical expected risk return $E\left[R_{t+1}\middle|\, s_H\right] > E\left[R_{t+1}\middle|\, s_L\right]$ and pro-cyclical risk-free rate $\left[r_t^f\middle|\, s_H\right] > \left[r_t^f\middle|\, s_L\right]$ and counter-cyclical risk premium

$$E\left[R_{t+1}\middle|\, s_H\right] - \left[r_t^f\middle|\, s_H\right] < E\left[R_{t+1}\middle|\, s_L\right] - \left[r_t^f\middle|\, s_L\right] \tag{4.6}$$

simultaneously.

In fact, sufficiently large distant between AiL and AiH namely $AiH - AiL$ also supports the above situations. In the extreme case, when $p_L \to 1$, $A_L \to 0$ and $i_L \to 0$, we have $\left[r_t^f\middle|\, s_t = s_L\right] \to 0$. In this case, there is no incentive for firms to invest and therefore the technology will not have enough development to pull the economy out of the low state BGP. Investors have low expectation on consumption growth. The risk-free rate approaches the zero lower bound.

To sum, in this simple two-states model, there are some key results. Firstly, expectations

[5] See appendix E

can be crucial to the investment and technology development and further can determine the long-run growth. Moreover, when the firms are pessimistic, the economy tend to be trapped in the low technology, low investment and slow growth state with close to 0 risk-free rate.

5 Calibration

In this section, I conduct a calibration to present the model findings in a quantitative sense. The calibration represents the model with the "threshold" assumption and sunspots. It is carried to match the economics fundamental variables moments in the data.

Given, A_L, A_H, p_L, p_H, I have closed form solution for risk-free rate r^f, risk premium RP, investment-GDP ratio $I/Y = i/A$ and growth rate of the economy $\phi(i) + 1 - \delta$ in both states.

$$E\left[R_{t+1}\middle| s_t = s_L\right] = \frac{1}{1-\beta}\phi'(i_L)\left[AiH - p_L(AiH - AiL)\right] \tag{5.1}$$

$$E\left[R_{t+1}\middle| s_t = s_H\right] = \frac{1}{1-\beta}\phi'(i_H)\left[AiL + p_H(AiH - AiL)\right] \tag{5.2}$$

$$\left[r_t^f\middle| s_t = s_L\right] = \frac{1}{1-\beta}\phi'(i_L)\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{5.3}$$

$$\left[r_t^f\middle| s_t = s_H\right] = \frac{1}{1-\beta}\phi'(i_H)\left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{5.4}$$

$$\left[\frac{I_t}{K_t}\middle| s_t = s_L\right] = i_L \tag{5.5}$$

$$\left[\frac{I_t}{K_t}\middle| s_t = s_H\right] = i_H \tag{5.6}$$

$$\left[\frac{I_t}{Y_t}\middle| s_t = s_L\right] = \frac{i_L}{A_L} \tag{5.7}$$

$$\left[\frac{I_t}{Y_t}\middle| s_t = s_H\right] = \frac{i_H}{A_H} \tag{5.8}$$

$$\left[\frac{Y_{t+1}}{Y_t}\middle| s_t = s_L\right] = \phi(i_L) + 1 - \delta \tag{5.9}$$

$$\left[\frac{Y_{t+1}}{Y_t}\middle| s_t = s_L\right] = \phi(i_H) + 1 - \delta \tag{5.10}$$

I consider two alternative functional forms for investment adjustment cost function $\phi(i)$ from Eberly & Wang (2009) and Gourio (2012) to perform a calibration. The former is in a

	α	Γ	θ	δ	β	A_L	A_H	p_L	p_H
Log	0.1	0.015	0.035	0.13	0.98	0.171	2.05	0.9935	0.998
Qud	–	1.7	0.025	0.13	0.98	0.11	0.14	0.9983	0.998

Table 1: Parameterization

log form given by

$$\phi(i) = \alpha + \Gamma \log\left(1 + \frac{i}{\theta}\right) \tag{5.11}$$

The latter is the frequently used quadratic form like

$$\phi(i) = i - \frac{\Gamma (i - \theta)^2}{2} \tag{5.12}$$

With the parameterisation shown in table 1, the model generates the moments in table 2 which can be compared with data. I use the macroeconomic data of US including investment-capital ratio, investment-GDP ratio, per capita GDP growth rate, treasury bond rate, and equity risk premium. All data are in real terms. Details of data description are in appendix H. Particularly, I use the period between 1992 to 2001 as the representation of high-growth state with a 2.28% per capita GDP growth. For the low-growth state, I use 2005 to 2014. In this period, on average, the growth rate of per capita GDP is 0.65%.

Parameterizations are set as shown in 1. The baseline value of the parameters are set according to related the value in the related literature. For the log-form, I mainly refer to Eberly & Wang (2009). The quadratic case, I borrow the parameters setup from Gourio (2012). The rest of the parameters are tunied to match the data moments.

As shown in table 2, overall, the investment adjustment function in the log form slightly performs better than the other. Figures in bold are those moments cannot match the data moments even in a rough sense. Since the model has a simple structure and the states are static, I do not expect more accuracy on the calibration. Roughly, the log-formed model can match the moments of investment-capital ratio $i = I/K$, GDP growth rate and counter-cyclical risk premium in both states. Especially, it captures the property of data that in the low-growth state risk-free rate drop dramatically to 0.9% and the risk premium is increasing since the risk return are relatively stable. However, there is calibration trade-offs. The log-form can not obtain proper investment-capital ratio and asset returns simultaneously. The 84% for i shows we need a very high accumulation of $A_H = 2.05$ to generate high growth. In terms of the quadratic form, it obtains the value for investment-capital ratio at around 10%. Nonetheless, it cannot offer an appropriate distance between technology A and i to generate

%	Calibration - Log		Calibration - Qud		Data of US	
	High	Low	High	Low	High	Low
$i = I/K$	**84**	5.3	17.5	12.7	11.5	10.4
I/Y	40.3	31.9	**86.3**	**83.8**	34.8	32
Growth Rate	1.8	**-1.6**	2.25	**-1.1**	2.28	0.65
Risk Free Rate	**2.0**	0.9	4.3	0.88	3.27	0.88
Risk Premium	**1.7**	5.6	**0.00**	**0.00**	3.03	5.11

Table 2: Calibration and Data

the investment-GDP ratio I/Y. Though it dose generates counter-cyclical risk premium, they are negligible.

Generally, the calibration shows the model can, to some extent, capture the risk premium behaviour shown in the data. An on-going work is to conduct a GMM estimation and statistic test. Another direction is introducing Epstein and Zin utility function to call for more parameters and study influence of the elasticity of intertemporal elasticity of substitution (IES). There are many works of literature have studies the recursive utility can provide some theoretical explanation of the behaviour of the risk-free rate. Introducing this preference might also improve the calibration of the model.

6 Conclusion Remarks

This paper is constructed to explore secular stagnation and its implications on asset pricing. I develop an AK model. It links firms investment to macroeconomic fundamentals. In this framework, the firm's unique purpose is the maximisation of its stock value. It is determined by two things which are 1) the fundamental namely the dividends payment and 2) the discount factor. Considering the first one and imaging a constant discount factor, the first balance faced by the firm is the intertemporal choice of dividends payment. Retaining investment for next period production will harm the current period dividends and support the next period dividends. Adding investment adjustment cost into the consideration will slightly complicate the situation and make the firm less willing to catch up with the technology accumulation. Further, since the discount factor is formed by investors intertemporal marginal rate of utility, the firm will consider the substitution effect on the asset prices. Again, the adjustment cost of investment plays an important role in this decision-making process. In all, the with the presence of investment adjustment cost, the relation between technology, investment and consumption growth might not be as straight forward as in conventional models.

With log utility and the AK production function, the baseline model reveals a link between technology and investment. Besides, the baseline model shows that though we have a monotonic relation between technology A and investment-capital ratio i, the growth of consumption is not in a simple linear relationship with technology A since there is investment adjustment cost for firm's investment. Additionally, the baseline model shows, with exogenous i.i.d technology, we have a counter-cyclical risk premium. The source of this phenomenon is the contradicted force in substitution effect and more importantly the convex adjustment cost of investment-capital ratio.

Further, I make an assumption which endogenizes the technology A. By doing that, the model immediately exhibits multiple equilibria with different macroeconomic fundamentals such as economic growth rate, risk-free rate, risk return and so forth. Later, I call for the sunspots which serve as a selecting device to capture and idea of beliefs is the main source to drive the economy shifting between BGPs. In general, the model can produce an arbitrarily long period of slow growth simply by adjusting the distribution of sunspots and expectation.

The model is simply constructed and can have closed form solution for the all the moments of macroeconomic variables introduced in the model. In the calibration , the model can capture the growth rate and risk premium relatively well but has a trade-off when includes the investment-capital ratio. A further direction of the paper is modifying the log utility function to recursive EZ framework to capture more information on the movement of the risk-free rate.

Appendix

A Convexity of Investment Adjustment Cost

Here I show that our restrictions on the capital accumulation function are consistent with the convex restriction on the adjustment cost function in the literature. Our capital accumulation condition (2.2) can be re-expressed as

$$K_{t+1} = I_t + (1-\delta) K_t - [i_t - \phi(i_t)] K_t \tag{A.1}$$

If the third term is zero, the current period investment is directly transformed into tomorrow's capital input without cost. Accordingly, we can treat the term in the bracket as adjustment cost of investment. Denote it as $Cst(i)$. Then our restrictions $1 > \phi'(i) > 0$ and $\phi''(i) < 0$ immediately lead to,

$$Cst'(i) = 1 - \phi'(i) > 0 \tag{A.2}$$

$$Cst''(i) = -\phi''(i) > 0 \tag{A.3}$$

i.e. our restrictions on $\phi(i)$ indicate adjustment cost is convex.

B Proof of Hayashi Proposition in My Model

Essentially what I need to prove is, in this model with the assumed functional form, $V_{K,t} = V_t/K_t$.

Proof. I start from the FOC for I_t,

$$E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right) V_{K_{t+1}}\right] = \frac{1}{\phi'(i_t)} \tag{B.1}$$

I use the envelope theorem to derive the FOC of capital K_t,

$$V_{Kt} = A_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right) V_{Kt+1}\right] [1 - \delta + \phi(i_t) - i\phi'(i_t)] \tag{B.2}$$

I multiply both sides by K_t, yield

$$V_{Kt}K_t = A_tK_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]\left[K_t\left((1-\delta)+\phi\left(i_t\right)\right) - i_t\phi'\left(i_t\right)K_t\right] \tag{B.3}$$

$$= A_tK_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}K_{t+1}\right] - E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]\phi'\left(i_t\right)I_t \tag{B.4}$$

$$= A_tK_t - I_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}K_{t+1}\right] \tag{B.5}$$

The third equality I use the FOC for I_t in (B.1). Equation (B.5) can be forward and iterated to obtain,

$$V_{K,t}K_t = E_t\left\{\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(\frac{\Lambda_\tau}{\Lambda_t}\right)\left(A_\tau K_\tau - I_\tau\right)\right\} \tag{B.6}$$

$$= V_t \tag{B.7}$$

Q.E.D. □

C Consumer's Problem

The representative consumer faces a standard infinite horizon utility maximization problem given by,

$$J_t = \underset{C}{Max} \quad E_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}U\left(C_t\right)\right] \tag{C.1}$$

subject to budget constrain

$$S_{t+1}P_t = S_t\left(P_t + D_t\right) - C_t \tag{C.2}$$

$$C_t > 0, \quad S_t > 0 \tag{C.3}$$

$$V_t > 0, \quad D_t > 0 \tag{C.4}$$

The Bellman equation can be written as

$$J_t\left(S_t\right) = \underset{C}{Max} \quad U\left(C_t\right) + \beta E_t\left(J_{t+1}\left(S_{t+1}\right)\right) \tag{C.5}$$

FOC of consumption C_t gives,

$$U'(C_t) = \beta E_t \left(\frac{J_{St+1}}{V_t} \right) \tag{C.6}$$

Envelope theorem offers

$$J_{St} = \beta E_t \left[J_{St+1} \left(\frac{P_t + D_t}{V_t} \right) \right] \tag{C.7}$$

I combine the two, yield

$$J_{St} = U'(C_t)(P_t + D_t) \tag{C.8}$$

Further, I forward 1 period and substitute back into (C.7). I obtain

$$U'(C_t)(P_t + D_t) = \beta E_t \left[U'(C_{t+1})(P_{t+1} + D_{t+1}) \left(\frac{P_t + D_t}{P_t} \right) \right] \tag{C.9}$$

$$U'(C_t) P_t = \beta E_t \left[U'(C_{t+1})(P_{t+1} + D_{t+1}) \right] \tag{C.10}$$

which is the Euler equation.

D Transversality Condition of Baseline Model

In the firms problem, the transversality condition is

$$\lim_{t \to \infty} \beta^t V_t = 0 \tag{D.1}$$

From 2.11 we have

$$V_t = K_t \left[A_t - i_t + \frac{\phi(i_t) + 1 - \delta}{\phi'(i_t)} \right] \tag{D.2}$$

In equilibrium, the term in the bracket is constant, given K_0, we have

$$\lim_{t \to \infty} \beta^t K_t \left[A_t - i_t + \frac{\phi(i_t) + 1 - \delta}{\phi'(i_t)} \right] = K_0 \left[A - i + \frac{\phi(i) + 1 - \delta}{\phi'(i)} \right] \lim_{t \to \infty} \left[\beta (\phi(i) + 1 - \delta) \right]^t \tag{D.3}$$

Accordingly, the transversality condition is

$$\beta\left(\phi\left(i\right)+1-\delta\right)<1$$

The consumer's problem has a transversality condition given by

$$\lim_{t\to\infty}\beta^{t}J_{t}=0$$

In the equilibrium, consumption constantly grow at rate $\phi\left(i\right)+1-\delta$, given C_t

$$J_{t}=E_{t}\sum_{\tau=t}^{\infty}\beta^{\tau-t}U\left(C_{t}\right) \tag{D.4}$$

With Log-uitlity,

$$J_{t}=\sum_{\tau=t}^{\infty}\beta^{\tau-t}\log\left[C_{t}\left(\phi\left(i\right)+1-\delta\right)^{\tau-}\right] \tag{D.5}$$

$$=\log C_{t}\sum_{\tau=t}^{\infty}\beta^{\tau-t}+\log\left(\phi\left(i\right)+1-\delta\right)\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(\tau-t\right) \tag{D.6}$$

$$=\frac{1}{1-\beta}\log C_{t}+\log\left(\phi\left(i\right)+1-\delta\right)\frac{\beta}{\left(\beta-1\right)^{2}} \tag{D.7}$$

Then, given C_0 and $0<\beta<1$,

$$\lim_{t\to\infty}\beta^{t}J_{t}=\frac{1}{1-\beta}\lim_{t\to\infty}\beta^{t}\log C_{t} \tag{D.8}$$

$$=\frac{1}{1-\beta}\lim_{t\to\infty}\beta^{t}\log\left[C_{0}\left(\phi\left(i\right)+1-\delta\right)^{t}\right] \tag{D.9}$$

$$=\frac{1}{1-\beta}\lim_{t\to\infty}\beta^{t}\log C_{0}+\frac{1}{1-\beta}\lim_{t\to\infty}\beta^{t}\log\left(\phi\left(i\right)+1-\delta\right)^{t} \tag{D.10}$$

$$=\frac{\log\left(\phi\left(i\right)+1-\delta\right)}{1-\beta}\lim_{t\to\infty}\beta^{t}t \tag{D.11}$$

$$=0 \tag{D.12}$$

E Root of the Solution in the Baseline Model

This section I show there is only root in the model solution equation in the baseline model. It is clear the equation (**??**) is non-linear for a given A in the defined range $A > i > 0$.

$$\frac{\phi(i) + 1 - \delta}{\phi'(i)(A - i)} = \frac{\beta}{1 - \beta} \tag{E.1}$$

Firstly, I consider the number of roots to this equation.

With the restriction of $A > i$, $\phi(i) > 0$, $\phi'(i) > 0$ and $\phi''(i) < 0$, we can write

$$(1 - \beta)[\phi(i) + 1 - \delta] - \beta\phi'(i)(A - i) = 0 \tag{E.2}$$

If I define the left hand side as $f(i)$, in the defined range $A > i > 0$, we have

$$\frac{\partial f}{\partial i} = (1 - \beta)\phi'(i) - \beta[\phi''(i)(A - i) - \phi'(i)] \tag{E.3}$$

$$\frac{\partial f}{\partial i} > 0 \tag{E.4}$$

Additionally, we have

$$f(0) = (1 - \beta)[\phi(0) + 1 - \delta] - \beta\phi'(0)A \tag{E.5}$$

$$f(A) = (1 - \beta)[\phi(A) + 1 - \delta] \tag{E.6}$$

Thus the sufficient condition grantees the uniqueness of the root is

$$A > \frac{(1 - \beta)[\phi(0) + 1 - \delta]}{\beta\phi'(0)} \tag{E.7}$$

Secondly, I derive the relation between A and i. Implicit function theorem offers

$$\frac{\partial i}{\partial A} = \frac{[\phi(i) + 1 - \delta]\phi'(i)}{\left[(\phi'(i))^2(A - i) - \phi''(i)[\phi(i) + 1 - \delta](A - i)\right] + [\phi(i) + 1 - \delta]\phi'(i)} \tag{E.8}$$

With the previous conditions, we have $1 > \partial i/\partial A > 0$. Accordingly, we have $A_H - i[A_H] > A_L - i[A_L]$.

F The Second Order Derivative of Consumption Growth with Respect to Technology

This section lay out the second order derivative of consumption growth C_{t+1}/C_t with respect to technology A_t.

$$\frac{\partial^2 \left(C_{t+1}/C_t\right)}{\partial A_t^2} = \partial \left(\frac{\beta \left[\phi' \left(i \left[A\right]\right)\right]^2}{\phi' \left(i \left[A\right]\right) - \beta \phi'' \left(i \left[A\right]\right) \left(A - i \left[A\right]\right)} \right) / \partial A \tag{F.1}$$

$$= \frac{\beta 2 \phi'' \left(i \left[A\right]\right) i' \left[A\right]}{\phi' \left(i \left[A\right]\right) - \beta \phi'' \left(i \left[A\right]\right) \left(A - i \left[A\right]\right)} - \tag{F.2}$$

$$\frac{\beta \left[\phi' \left(i \left[A\right]\right)\right]^2 \left\{\phi'' \left(i \left[A\right]\right) i' \left[A\right] - \beta \phi''' \left(i \left[A\right]\right) i' \left[A\right] \left(A - i \left[A\right]\right) - \beta \phi'' \left(i \left[A\right]\right) \left(1 - i' \left[A\right]\right)\right\}}{\left[\phi' \left(i \left[A\right]\right) - \beta \phi'' \left(i \left[A\right]\right) \left(A - i \left[A\right]\right)\right]^2} \tag{F.3}$$

where $i' \left[A\right]$ is derived in proposition 2.

G Risky Asset Return and Risk Free Rate

This section in the Appendix I derive the return for the risky asset R_{t+1} the risk-free rate r_t^f. I use the following conditions derived in the baseline model.

$$K_{t+1} = \left(1 - \delta\right) K_t + \phi \left(i_t\right) K_t \tag{G.1}$$

$$\frac{V \left(K_t\right)}{K_t} = A_t - i_t + \frac{\phi \left(i_t\right) + 1 - \delta}{\phi' \left(i_t\right)} \tag{G.2}$$

$$\frac{\phi \left(i_t\right) + 1 - \delta}{\phi' \left(i_t\right) \left(A_t - i_t\right)} = \frac{\beta}{1 - \beta} \tag{G.3}$$

I start from the included dividends asset return,

$$R_{t+1}^{IND} = \frac{V_{t+1}}{V_t} \tag{G.4}$$

$$= \frac{V_{t+1}/K_{t+1}}{V_t/K_t}\frac{K_{t+1}}{K_t} \tag{G.5}$$

$$= \frac{A_{t+1} - i_{t+1} + \frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})}}{A_t - i_t + \frac{\phi(i_t)+1-\delta}{\phi'(i_t)}} \left(\phi\left(i_t\right) + 1 - \delta\right) \tag{G.6}$$

$$= \frac{A_{t+1} - i_{t+1} + \left(A_{t+1} - i_{t+1}\right)\frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})(A_{t+1}-i_{t+1})}}{A_t - i_t + \left(A_t - i_t\right)\frac{\phi(i_t)+1-\delta}{\phi'(i_t)(A_t-i_t)}} \left(\phi\left(i_t\right) + 1 - \delta\right) \tag{G.7}$$

$$= \frac{\left(A_{t+1} - i_{t+1}\right)}{\left(A_t - i_t\right)}\left[\phi\left(i_t\right) + 1 - \delta\right] \tag{G.8}$$

$$= \left(A_{t+1} - i_{t+1}\right)\phi'\left(i_t\right)\frac{\left[\phi\left(i_t\right) + 1 - \delta\right]}{\phi'\left(i_t\right)\left(A_t - i_t\right)} \tag{G.9}$$

$$= \left(A_{t+1} - i_{t+1}\right)\phi'\left(i_t\right)\frac{\beta}{1-\beta} \tag{G.10}$$

Since V_t is the included dividend price we derive the ex-dividend return as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} \tag{G.11}$$

$$= \frac{V_{t+1}}{V_t - D_t} \tag{G.12}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{D_t}{V_{t+1}}\right]^{-1} \tag{G.13}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{D_t/K_t}{V_{t+1}/K_{t+1}} \frac{K_t}{K_{t+1}}\right]^{-1} \tag{G.14}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{A_t - i_t}{A_{t+1} - i_{t+1} + \frac{\phi(i_{t+1})+1-\delta}{\phi'(i_{t+1})}} \frac{1}{\phi\left(i_t\right) + 1 - \delta}\right]^{-1} \tag{G.15}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{1}{A_{t+1} - i_{t+1} + \left(A_{t+1} - i_{t+1}\right) \frac{\beta}{1-\beta}} \frac{1}{\left(\frac{\beta}{1-\beta}\right) \phi'\left(i_t\right)}\right]^{-1} \tag{G.16}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{1}{\left(A_{t+1} - i_{t+1}\right) \left(\frac{1}{1-\beta}\right)} \frac{1}{\left(\frac{\beta}{1-\beta}\right) \phi'\left(i_t\right)}\right]^{-1} \tag{G.17}$$

$$= \left[\frac{1}{\left(A_{t+1} - i_{t+1}\right) \phi'\left(i_t\right) \frac{\beta}{1-\beta}} - \frac{1}{\left(A_{t+1} - i_{t+1}\right) \left(\frac{1}{1-\beta}\right)} \frac{1}{\left(\frac{\beta}{1-\beta}\right) \phi'\left(i_t\right)}\right]^{-1} \tag{G.18}$$

$$= \left\{\frac{1}{\left(A_{t+1} - i_{t+1}\right) \phi'\left(i_t\right) \frac{\beta}{1-\beta}} \left[1 - \frac{1}{\left(\frac{1}{1-\beta}\right)}\right]\right\}^{-1} \tag{G.19}$$

$$= \left(A_{t+1} - i_{t+1}\right) \frac{\phi'\left(i_t\right)}{1 - \beta} \tag{G.20}$$

The capital accumulation condition (G.1) and the solution for marginal q (G.2) are used in equality (G.6). The rest of the equality I repeatedly use the solution condition (G.3) for BGP level of i.

One can check the relation indicated by the Euler equation,

$$1 = E_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t} \cdot R_{t+1}\right] \tag{G.21}$$

For the risk-free rate, by definition it is the inverse of expectation of SDF as,

$$r_t^f = \frac{1}{E\left(\beta \frac{\Lambda_{t+1}}{\Lambda_t}\right)} \tag{G.22}$$

$$= \left[E\left(\beta \cdot \frac{A_t - i_t}{A_{t+1} - i_{t+1}} \frac{K_t}{K_{t+1}}\right)\right]^{-1} \tag{G.23}$$

$$= \left[E\left(\frac{1}{A_{t+1} - i_{t+1}}\right) \beta \left(\frac{A_t - i_t}{\phi\left(i_t\right) + 1 - \delta}\right)\right]^{-1} \tag{G.24}$$

$$= \left[E\left(\frac{1}{A_{t+1} - i_{t+1}}\right) \beta \left(\frac{1 - \beta}{\beta \phi'\left(i_t\right)}\right)\right]^{-1} \tag{G.25}$$

$$= \left[E\left(\frac{1}{A_{t+1} - i_{t+1}}\right)\right]^{-1} \frac{\phi'\left(i_t\right)}{1 - \beta} \tag{G.26}$$

The capital accumulation condition (G.1) is used in the third equality and the solution condition(G.3) for of i is used in the forth equality.

H Data Description

I use the 1992 - 2001 as the fast growth state and 2005 to 2014 as the slow growth state. I collect the data from Fred Economic Data[6] and Quandl[7]. All data are quarterly data and in real terms. GDP growths are in per capital. The Risk premium is calculated by the return on stock index S&P 500 neglecting the 10 treasury bond rate.

[6] research.stlouisfed.org

[7] www.quandl.com

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